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1996 J. Phys. A: Math. Gen. 29 5989

(http://iopscience.iop.org/0305-4470/29/18/027)

J. Phys. A: Math. Gen. **29** (1996) 5989–6001. Printed in the UK

Dromion-like structures in a $(3+1)$-dimensional KdV-type equation

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Received 11 December 1995, in final form 3 June 1996

Abstract. Dromion structures have been found in many $(2+1)$-dimensional models. The similar structure in $(3+1)$ dimensions is studied in this paper for a KdV-type equation, $w_t + 6w_x w_y + w_{xxy} + w_{xxxxz} + 60w_x^2 w_z + 10w_z w_{xxx} + 20w_x w_{xxz} = 0$. Starting from the bilinear form of the model, we found that there are five types of multi-dromion solutions for its potentials, say, $v_1 \equiv w_y$. The first type of multi-dromion solution is driven by multi-camber solitons with one of them being non-parallel to the x-axis. The second, third and fourth types of multi-dromion solutions are driven by two camber solitons and one, two and three sets of parallel plane solitons, respectively. The fifth type of multi-dromion solution is driven by one set of parallel plane solitons, one set of camber solitons which are parallel to the x-axis and one camber soliton which is non-parallel to the x-axis. Phase shifts may be involved in the interactions among the multi-dromions for the first and fifth types of solutions. A single dromion solution of the model may possess quite a free shape. For instance, the point-like dromions, ring-type dromions, extended and sharp dromions and oscillatory dromions can be obtained by selecting some arbitrary functions appropriately.

1. Introduction

Since the soliton phenomena were first observed by Scott Russell in 1834 [1] and the KdV equation was solved by the inverse scattering method by Gardner *et al* in 1967 [2], the study of solitons and the related issue of the construction of solutions to a wide class of nonlinear equations has become one of the most exciting and extremely active areas of research investigation. Early in the study of soliton theory, the main interests of scientists were restricted to the $(1+1)$-dimensional cases because of the difficulty of finding the physically significant high-dimensional solutions which are localized in all directions. Recently, the study of soliton-like structures in high dimensions has attracted much more attention. In particular, for some $(2+1)$-dimensional integrable models such as the Davey–Stewartson (DS) [3], Kadomtsev–Petviashvilli (KP) [4] and Nizhnik–Novikov–Veselov (NNV) [5] models, some types of solutions (called dromions) which are localized in all directions are found by using some different approaches. Generally, a dromion solution occurs at the cross point of two non-parallel line solitons. A line soliton is finite on an infinitely long straight line while it exponentially decays in other directions. For the DS and NNV equations, the dromion solutions can be obtained from two perpendicular line ghost solitons

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[3, 5] while for the KP equation the dromion solution can be driven by non-perpendicular line solitons [4]. More recently, we found that the dromions in $(2+1)$-dimensions may possess much more abundant structures [6, 7] because the $(2+1)$-dimensional integrable models possess infinite-dimensional Kac–Moody–Virasoro type symmetries which contain some arbitrary functions [8, 9]. For instance, for a $(2+1)$-dimensional KdV equation [6], the multi-dromion solutions can be driven, not only by some perpendicular line and non-perpendicular line ghost solitons, but also by curved line ghost solitons. A curved line soliton is defined as a solution which is finite on a curved line and exponentially decays away from the curve.

Now two interesting questions are: Can we find a $(3+1)$-dimensional dromion-like structure which is localized in all directions? What kind of different properties of a $(3+1)$-dimensional dromion (if there exists) will there be?

In this paper we study only the dromion-like structures for a non-integrable KdV-type toy model:

$$w_t + 6w_x w_y + w_{xxy} + w_{xxxxz} + 60w_x^2 w_z + 10w_z w_{xxx} + 20w_x w_{xxz} = 0. \quad (1)$$

When we restrict w to being z-independent, equation (1) reduces to

$$w_t + 6w_x w_y + w_{xxy} = 0 \quad (2)$$

which is a potential form of the equation ($u = \frac{3}{2}w_x,\ b = 1$)

$$u_t + bu_{xxy} + 4buu_y + 4bu_x \partial_x^{-1} u_y = 0 \quad (3)$$

studied by Radha and Lakshmanan [10].

Equations (3) and (2) will reduce to the well known KdV and potential KdV equations, respectively, for $y = x$. On the other hand, if w is y-independent and $z = x$, equation (1) becomes the usual potential form of the Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK) equation.

It is known that both the potential KdV equation and the potential CDGSK equation can be bilinearized [11] by means of the transformation

$$w = (\ln \phi)_x. \quad (4)$$

There are several effective methods to bilinearize a nonlinear partial differential equation, say, the Painlevé singularity analysis [12] and direct bilinearization methods [13]. Using any one of these methods, one can easily find that the same transformation (4) will change the $(3+1)$-dimensional KdV equation (1) to a bilinear form,

$$(D_x D_t + D_x^3 D_y + D_x^5 D_z)\phi \cdot \phi = 0 \quad (5)$$

where the bilinear D-operators are defined by [11]

$$D_{\boldsymbol{x}}\phi \cdot \phi = (\partial_{\boldsymbol{x}} - \partial_{\boldsymbol{x}'})\phi(x, y, z, t) \cdot \phi(x', y', z', t')|_{(x,y,z,t)=(x',y',z',t')}$$

with $\boldsymbol{x} = (x, y, z, t)$. If we use the Painlevé singularity analysis to bilinearize the model we can prove that equation (1) does not possess the Painlevé property at the same time.

The bilinear method [11] has always been very useful for constructing soliton solutions and has also shown its power in the study of dromion solutions [10]. In the next section, we will analyse some interesting dromion structures of equation (1) by solving the bilinear equation of (1). Similar to the dromion structure of the $(2+1)$-dimensional equation (3), the dromion solutions exist for some potentials of (1), say, $v_1 \equiv w_y$ and $v_2 \equiv w_z$, instead of the fields w and $u = w_x$. The result shows us that dromion structures in $(3+1)$-dimensional cases are much more abundant than those in $(2+1)$-dimensional cases. For the KdV-type equation (1), there are five types of multi-dromion solution. For a single dromion solution, it may have quite a free shape. For instance, some ring types of dromion which are finite only on a closed curve are allowed while the ring dromion solutions have not yet been found in $(2+1)$ dimensions.

2. Dromion-like structures

To construct the dromion-like structure of (1), the simplest way is to solve its bilinear form (5) by using a power series such as

$$\phi = 1 + \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \epsilon^3\phi^{(3)} + \cdots \tag{6}$$

where ϵ is a small parameter. Substituting (6) into (5) and comparing the coefficients of various powers of ϵ, we obtain the following sets of linear equations:

$$\epsilon : \phi^{(1)}_{xt} + \phi^{(1)}_{xxxy} + \phi^{(1)}_{xxxxxz} = 0 \tag{7}$$

$$\epsilon^2 : \phi^{(2)}_{xt} + \phi^{(2)}_{xxxy} + \phi^{(2)}_{xxxxxz} = \tfrac{1}{2}(D_xD_t + D_x^3D_y + D_x^5D_z)\phi^{(1)} \cdot \phi^{(1)} \tag{8}$$

etc.

Solving equation (7) we have

$$\phi^{(1)} = \sum_{i=1}^{N} \exp(k_i x + f_i(\xi_{i1}, \xi_{i2}, \ldots, \xi_{iM})) + h(y, z, t) \tag{9}$$

where f_i and h are arbitrary functions of the indicated variables and

$$\xi_{ij} = P_{ij}y + Q_{ij}z - (k_i^2P_{ij} + k_i^4Q_{ij})t + x_{ij} \qquad (i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M) \tag{10}$$

with P_{ij}, Q_{ij} and x_{ij} being arbitrary constants.

Similar to $(2+1)$-dimensional cases, some arbitrary functions f_i and h have been included in the solution of (7). Differently, now the number of the arguments of arbitrary functions f_i is an arbitrary positive integer M.

Substituting equation (9) with (10) into (8) etc, the solutions, $\phi^{(j)} (j \geqslant 2)$, can be determined recursively.

2.1. $N = 1$ case

To get 'one' soliton solution, we can take $N = 1$ in (9). After substituting (9) with $N = 1$ into (8) etc, we find that if the arbitrary function h is also fixed as an arbitrary function of ξ_{1j} shown by equation (10), the higher-order component $\phi^{(j)}$ can be taken as zero for all $j \geqslant 2$. The field w reads

$$w = (\ln\phi)_x = \frac{k_1 \exp(k_1x + g_1)}{1 + h + \exp(k_1x + g_1)} = \frac{1}{2}\left(1 + \tanh\frac{1}{2}(k_1x + G)\right) \tag{11}$$

where

$$G \equiv G(\xi_{11}, \xi_{12}, \ldots, \xi_{1M}) \equiv g_1 - \ln(1 + h). \tag{12}$$

For the field w two arbitrary functions g and h are degenerated to one arbitrary function G.

It is clearly seen that as in the $(2+1)$-dimensional case ($w_z = 0$ for (1)), the field w is not exponentially localized in all directions. In the $(2+1)$-dimensional case, although the physical field does not exponentially decay in all directions, one of its potentials ($v \equiv w_y$) decays in all directions [10, 7]. Similarly, we consider the properties of some potentials of the $(3+1)$-dimensional model (1), say, $u \equiv w_x$, $v_1 \equiv w_y$ and $v_2 \equiv w_z$. From equation (11), we have

$$u = w_x = \tfrac{1}{4}k_1^4 \operatorname{sech}^2 \tfrac{1}{2}(k_1 x + G) \tag{13}$$

$$v_1 = w_y = \tfrac{1}{4}k_1 \sum_{i=1}^{M} P_{1i} G_{\xi_1 i} \operatorname{sech}^2 \tfrac{1}{2}(k_1 x + G) \tag{14}$$

$$v_2 = w_z = \tfrac{1}{4}k_1 \sum_{i=1}^{M} Q_{1i} G_{\xi_1 i} \operatorname{sech}^2 \tfrac{1}{2}(k_1 x + G). \tag{15}$$

To understand the meanings of the solutions (11) and (13)–(15), we first discuss some special cases.

(i) *Single camber soliton solutions.* For the field w, solution (11) shows that there is a centre curved surface,

$$S \equiv k_1 x + G(\xi_{11}, \xi_{12}, \ldots, \xi_{1M}) = 0. \tag{16}$$

Apart from this centre camber, the field w exponentially tends to two *different* values, 1 for $S \to +\infty$ and 0 for $S \to -\infty$. We call this type of solution the camber kink for simplicity.

For the potential u, solution (13) is finite on the camber (16) and decays exponentially away from the camber. We call this type of solution the camber soliton or camber solitary wave.

It is more interesting because, as in the $(2+1)$-dimensional case [7, 10], although the solitary waves for fields u and w are not localized in all directions, their potentials may exponentially decay in all directions. For the potential v_1 (or v_2), the structure of the soliton solution (14) (or (15)) is much more abundant.

(ii) *Single point dromion solutions.* For the potential v_1, if the arbitrary function G is fixed such that the following equation is satisfied

$$\sum_{i=1}^{M} P_{1i} G_{\xi_{1i}} = A_1 \operatorname{sech}^{n_1}\xi_{11} \operatorname{sech}^{n_2}\xi_{12} \equiv H_1 \tag{17}$$

where $n_1 > 0, n_2 > 0$, and A_1 is an arbitrary constant, we can get a single $(3+1)$-dimensional dromion-like solution driven by two plane solitons and one camber soliton:

$$v_1 = \tfrac{1}{4}k_1 A_1 (\operatorname{sech}^{n_1}\xi_{11})(\operatorname{sech}^{n_2}\xi_{12})(\operatorname{sech}^2 \tfrac{1}{2}(k_1 x + G)). \tag{18}$$

For $M = 2$, equation (17) can be integrated. After integrating equation (17) with $M = 2$, the function G in (18) can be written as

$$G = \frac{A_1}{2P_{11}P_{12}} \int^{P_{12}\xi_{11}+P_{11}\xi_{12}} \operatorname{sech}^{n_1}\left(\frac{1}{2P_{12}}(\eta+\theta)\right) \operatorname{sech}^{n_2}\left(\frac{1}{2P_{11}}(\eta-\theta)\right) \mathrm{d}\eta + f_0(\theta) \tag{19}$$

$$\theta \equiv P_{12}\xi_{11} - P_{11}\xi_{12}. \tag{20}$$

It is clearly seen that the soliton is localized in all directions and located at the cross point of the two planes (which are parallel to the x-axis), $\xi_{11} = 0, \xi_{12} = 0$, and one camber (16) with G being given by (19), i.e. the space position of the soliton is determined by

$$\xi_{11} = 0 \qquad \xi_{12} = 0 \qquad S = 0. \tag{21}$$

The dromion-like structures for the potential v_2 are almost the same as those of the potential v_1 after replacing P_{1i} by Q_{1i} in (17). So we discuss only the properties of v_1 later.

Actually, two plane solitons in equation (19) can be replaced by two camber solitons which are parallel to the x-axis because the arguments ξ_{11} and ξ_{12} in the right-hand side of (17) can be replaced by two arbitrary functions:

$$\xi_{11} \to h_1(\xi_{11}, \xi_{12}, \ldots, \xi_{1M}) \equiv h_1, \xi_{12} \to h_2(\xi_{11}, \xi_{12}, \ldots, \xi_{1M}) \equiv h_2. \qquad (22)$$

That is to say, a $(3+1)$-dimensional dromion which decays in all directions can be driven not only by two plane ($\xi_{11} = 0$, $\xi_{12} = 0$) solitons and one camber ($S = 0$) soliton but also by three camber solitons if we select h_1 and h_2 such that the equation system

$$h_1 = 0 \qquad h_2 = 0 \qquad S = 0 \qquad (23)$$

possesses a unique solution for the space variables (x, y, z). Two camber ($h_1 = 0, h_2 = 0$) solitons are parallel to the x-axis while the other one ($S = 0$) is not.

In $(2+1)$ dimensions, a dromion solution is driven by two line (curved line or straight line) solitons. The interaction between two line solitons (finite at an infinitely long line) makes two line solitons disappear (become ghost solitons) and a dromion survives at the cross point of two line solitons. In $(3+1)$ dimensions, a point-like dromion solution is driven by three surface (camber or plane) solitons. The interaction among three camber solitons survives a point-like dromion which is located at the cross point of three surfaces while the original camber solitons become ghosts.

From equation (14) we know that if the function G is selected to be linear in the variables y and z (i.e. ξ_{1i}), we cannot get a dromion-like solution. In other words, we cannot get a $(3+1)$-dimensional soliton-like solution which is localized in all directions driven by three plane solitons by selecting the arbitrary function G. At least one camber soliton is necessary to construct a $(3+1)$-dimensional dromion-like solution from equation (14).

(iii) *Single ring dromion solutions.* In $(2+1)$-dimensional cases [3–7, 10], we have not yet found a ring-type soliton solution. We call a solution a ring soliton to mean that a solution is finite on a closed curve and decays away from the curve. It is quite interesting that, for the $(3+1)$-dimensional model (1), a ring soliton can be constructed easily by selecting the arbitrary function G appropriately, say,

$$\sum_{i=1}^{M} P_{1i} G_{\xi_{1i}} = A_2 \operatorname{sech}^n h(\xi_{11}, \xi_{12}, \ldots, \xi_{1M}) \equiv H_2 \qquad (24)$$

where the camber

$$h \equiv h(\xi_{11}, \xi_{12}, \ldots, \xi_{1M}) = 0 \qquad (M \geqslant 2) \qquad (25)$$

is the surface of a cylinder. In this case, we get a ring dromion solution:

$$v_1 = \tfrac{1}{4} k_1 A_2 (\operatorname{sech}^n h)(\operatorname{sech}^2 \tfrac{1}{2}(k_1 x + G)). \qquad (26)$$

That is to say, a ring dromion is driven by a cylinder ($h = 0$) soliton which is parallel to the x-axis and a camber ($S = 0$) soliton. The ring dromion is located at the intersection of the cylinder and the camber.

The simplest selection of h in (26) for a ring dromion solution is

$$h = \sum_{i=1}^{M} a_i \xi_{1i}^2 - C^2 \qquad (M \geqslant 2) \qquad (27)$$

with arbitrary constants a_i and C. For $M = 2$ and h being given by equation (27), the function G in (26) reads

$$G = \frac{A_2}{2P_{11}P_{12}} \int^{P_{12}\xi_{11} + P_{11}\xi_{12}} \operatorname{sech}^n \left(\frac{a_1}{4P_{12}^2} (\eta + \theta)^2 + \frac{a_2}{4P_{11}^2} (\eta - \theta)^2 \right) \mathrm{d}\eta + f_0(\theta) \qquad (28)$$

where $f_0(\theta)$ is an arbitrary function of θ and θ is given by equation (20).

(iv) *Extended and sharp dromions.* Because G is an arbitrary function in (14), the dromions may decay much slower than an exponential or much faster than an exponential in the y- and z-directions. For instance, if we select that the function G satisfies

$$\sum_{i=1}^{M} P_{1i} G_{\xi_{1i}} = \left(\sum_{i=1}^{M}\sum_{j=0}^{2N_1} a_{ij}\xi_{1i}^{j}\right)^{-1}\left(\sum_{i=1}^{M}\sum_{j=0}^{2N_2} b_{ij}\xi_{1i}^{j}\right)^{-1} \equiv g_1 g_2 \equiv H_3 \tag{29}$$

with g_1 and g_2 being analytical with respect to (x, y, z, t) and a_{ij} and b_{ij} being arbitrary constants, we get an extended point dromion (or ring dromion for $g_2 = 1$) solution which decays exponentially in the x-direction and decays rationally in the y- and z-directions. If we select the function G as

$$\sum_{i=1}^{M} P_{1i} G_{\xi_{1i}} = A\left(\operatorname{sech}^{n_1}\left(\cosh\sum_{i=1}^{M}\sum_{j=0}^{N_1} a_{ij}\xi_{1i}^{j}\right)\right)\left(\operatorname{sech}^{n_2}\left(\cosh\sum_{i=1}^{M}\sum_{j=0}^{N_1} a_{ij}\xi_{1i}^{j}\right)\right) \equiv H_4 \tag{30}$$

we get a sharp point or ring dromion solution which decays very much quicker than an exponential in the y- and z-directions.

(v) *Oscillatory dromions.* If some oscillatory functions are included in the arbitrary function G, say,

$$\sum_{i=1}^{M} P_{1i} G_{\xi_{1i}} = A_5 \sin(h(\xi_{11}, \xi_{12}, \ldots, \xi_{1M}))H_j \equiv H_5 \tag{31}$$

where H_j, $(j = 1–4)$ are defined in equations (17), (24), (29) and (30), respectively, any type of dromion (point dromion, ring dromion, extended and sharp dromion) may have an oscillatory structure both in amplitude and in phase.

(vi) *The first type of multi-dromion.* Generally, the dromion-like solution (14) exhibits the first type of multi-dromion structure. Any number of point dromions, ring dromions, extend dromions, sharp dromions and oscillatory dromions can be combined in some different ways because of the arbitrariness of the function G. The first simple way is combining them almost linearly,

$$\sum_{i=1}^{M} P_{1i} G_{\xi_{1i}} = \sum_{i=1}^{5}\sum_{j=1}^{J_{ij}} A_{ij} H_{ij} \equiv H_6 \tag{32}$$

where J_{ij} are arbitrary integers, A_{ij} are arbitrary constants and H_{ij} possess the form of H_i for all j. The corresponding form for the potential v_1 reads

$$v_1 = \tfrac{1}{4}k_1 H_6 \operatorname{sech}^2 \tfrac{1}{2}(k_1 x + G). \tag{33}$$

If the multi-soliton solutions are combined as in equations (32) and (33), we know that all the effects of the nonlinear interactions among dromions are included only in one camber soliton which is not parallel to the x-axis. If we are sitting on the camber, $S(\equiv k_1 + G = 0)$, to observe the interactions of the multi-dromions shown by (33), we will see that the dromions interact linearly. In other words, there is no phase shift involved in the interactions of this kind of dromion.

The second simple way to construct the multi-dromion solution from the solution (14) is to select

$$\sum_{i=1}^{M} P_{1i} G_{\xi_{1i}} = H_7(X_1, X_2, \ldots, X_D, T) \equiv H_7 \tag{34}$$

as a multi-soliton solution of any $(D+1)$-dimensional integrable model (we call this model a seed model), say, $(1+1)$-dimensional KdV and/or $(2+1)$-dimensional KP equations, with 'spacetime' variables $X_i \equiv X_i(\xi_{11}, \xi_{12}, \ldots, \xi_{1M})$, $i = 1, 2, \ldots, D$, and $T \equiv T(\xi_{11}, \xi_{12}, \ldots, \xi_{1M})$ being arbitrary functions of ξ_{1j}, $j = 1, 2, \ldots, M$. Now if we are again sitting on the camber, $S(\equiv k_1 + G = 0)$, to observe the interactions of the multi-dromions constructed from the seed models, we will see that the interaction among dromions looks like that of the seed models.

2.2. N=2 case

From the discussions of the last subsection, we know that the first type of multi-dromion solution is driven by only one camber ghost soliton which is non-parallel to the x-axis while all the other ghost camber solitons are parallel to the x-axis. Now an interesting question is: Can we find other types of dromion solutions which are driven by more camber solitons which are non-parallel to the x-axis? To find these kinds of solutions, we should take N in equation (9) to be larger than one.

Substituting equation (9) with $N = 2$ into (8) etc, we find that ϕ may have the following form,

$$\phi = h + \exp(k_1 x + g_1) + \exp(k_2 x + g_2) + h_1 \exp(k_1 x + k_2 x + g_1 + g_2) \tag{35}$$

if four functions $h \equiv h(y, z, t)$, $h_1 \equiv h_1(y, z, t)$, $g_1 \equiv g_1(y, z, t)$ and $g_2 \equiv g_2(y, z, t)$ satisfy some constraints. Here are some special examples.

(vii) *The second type of multi-dromion.* If the functions h, g_1 and g_2 are all restricted to be arbitrary functions of a single variable

$$\xi = z - (k_1^2 + k_2^2)y + k_1^2 k_2^2 t \tag{36}$$

we can take $h = 1$ without loss of generality and then the interaction factor between two camber solitons, h_1, is related to g_1 and g_2 by

$$h_1 = \exp(-g_1 - g_2) \times\left(C + \int^{\xi} \frac{(k_1 - k_2)(2k_2 - k_1)(k_2 - 2k_1)(g_{1\xi'} - g_{2\xi'})}{(k_1 + k_2)(2k_2 + k_1)(2k_1 + k_2)} \exp(g_1 + g_2)\, \mathrm{d}\xi'\right) \tag{37}$$

for

$$k_2 \neq -k_1 \qquad k_2 \neq -2k_1 \qquad k_1 \neq -2k_2 \tag{38}$$

where C is an arbitrary constant. The corresponding soliton solutions for the fields w, u and the potential v_1 read

$$w = \frac{k_1 J_1 + k_2 J_2 + (k_1 + k_2) h_1 J_1 J_2}{1 + J_1 + J_2 + h_1 J_1 J_2} \qquad (J_i \equiv \exp(k_i x + g_i), i = 1, 2) \tag{39}$$

$$u = \frac{(h_1(k_1 + k_2)^2 + (k_1 - k_2)^2) J_1 J_2 + k_1^2 J_1 + k_2^2 J_2 + h_1 k_1^2 J_1 J_2^2 + h_1 k_2^2 J_2 J_1^2}{(1 + J_1 + J_2 + h_1 J_1 J_2)^2} \tag{40}$$

$$\begin{aligned} v_1 = \frac{-(k_1^2 + k_2^2)}{(1 + J_1 + J_2 + h_1 J_1 J_2)^2} &\{g_{1\xi}[k_1 J_1 + (k_1 - k_2 + h_1(k_1 + k_2 + k_1 J_2)) J_2 J_1] \\ &+ g_{2\xi}[k_2 J_2 + (k_2 - k_1 + h_1(k_2 + k_1 + k_2 J_1)) J_2 J_1] \\ &+ h_{1\xi} J_1 J_2 (k_1 + k_2 + k_1 J_2 + k_2 J_1)\}. \end{aligned} \tag{41}$$

In this case, after selecting g_1, g_2 and h_1 appropriately for the potential v_1, we can get the second type of multi-dromion driven by multi-parallel plane solitons (which are determined by the functions g_1, g_2 and h_1) and two camber solitons (which are non-parallel to the x-axis and described by the curved surfaces $S_1(\equiv k_1x+g_1=0)$ and $S_2(\equiv k_2x+g_2=0)$). Because these parallel plane solitons move at the *same* speed perpendicular to the x-axis, the dromions cannot meet each other. Different from the first type of multi-dromion solution, the ring type of dromion is not allowed for the second type of multi-dromion solution because only one argument is included in the functions g_1, g_2 and h_1.

(viii) *The third type of multi-dromion.* If h and h_1 do not possess the same argument as those of the functions g_1 and g_2, we find that h and h_1 can be taken as

$$h=1 \qquad h_1=a_{12}=\text{constant}. \tag{42}$$

In this case, we have two subcases.

(a) Multi-dromions. The functions $g_1=g_1(\xi_1)$ and $g_2=g_2(\xi_2)$ are two different arbitrary functions of the arguments

$$\xi_1=P_1y+Q_1z-(k_1^2P_1+k_1^4Q_1)t+x_{10} \tag{43}$$

and

$$\xi_2=P_2y+Q_2z-(k_2^2P_2+k_2^4Q_2)t+x_{20} \tag{44}$$

where the constants P_1, P_2, Q_1 and Q_2 are related to the constants k_1, k_2 and a_{12} by

$$P_1=(a_{12}-1)(4k_1^4k_2+10k_1^2k_2^3+k_2^5)+(a_{12}+1)(10k_1^3k_2^3+5k_1k_2^4) \tag{45}$$

$$Q_1=2(1-a_{12})k_1^2k_2-3(a_{12}+1)k_1k_2^2)+(1-a_{12})k_2^3 \tag{46}$$

$$P_2=(a_{12}-1)(4k_2^4k_1+10k_2^2k_1^3+k_1^5)+(a_{12}+1)(10k_2^3k_1^3+5k_2k_1^4) \tag{47}$$

$$Q_2=2(1-a_{12})k_2^2k_1-3(a_{12}+1)k_2k_1^2)+(1-a_{12})k_1^3. \tag{48}$$

The corresponding fields w, u and v_1 are

$$w=\frac{k_1J_1+k_2J_2+(k_1+k_2)a_{12}J_1J_2}{1+J_1+J_2+a_{12}J_1J_2} \qquad (J_i\equiv\exp(k_ix+g_i),\, i=1,2) \tag{49}$$

$$u=\frac{(a_{12}(k_1+k_2)^2+(k_1-k_2)^2)J_1J_2+k_1^2J_1+k_2^2J_2+a_{12}k_1^2J_1J_2^2+a_{12}k_2^2J_2J_1^2}{(1+J_1+J_2+a_{12}J_1J_2)^2} \tag{50}$$

$$\begin{aligned}v_1=&\frac{P_1g_{1\xi_1}[k_1J_1+(k_1-k_2+a_{12}(k_1+k_2+k_1J_2))J_2J_1]}{(1+J_1+J_2+a_{12}J_1J_2)^2}\\&+\frac{P_2g_{2\xi_2}[k_2J_2+(k_2-k_1+a_{12}(k_2+k_1+k_2J_1))J_2J_1]}{(1+J_1+J_2+a_{12}J_1J_2)^2}.\end{aligned} \tag{51}$$

For the potential v_1, the third type of multi-dromion solutions can be obtained from (51) by selecting the functions g_1 and g_2 suitably, say,

$$g_{i\xi}=\sum_{m=1}^{M}A_m\,\text{sech}^{n_m}(B_m\xi_i+\xi_{0m}) \qquad (i=1,2)$$

with arbitrary constants A_m, B_m, ξ_{0m} and $n_m>0$. This type of multi-dromion solution is driven by two sets of plane solitons and two camber solitons. Although the planes in the same set are parallel, two sets of plane solitons are not parallel because the arguments of the functions g_1 and g_2 are different. Two camber solitons are not parallel to the x-axis. Similar to the second type of multi-dromion solution, the ring type of dromion is also not allowed for this kind of solution because every one of the functions g_1 and g_2 is only a function of a single variable. In this case, there is also no phase shift among the interactions of the multi-dromions. If we are sitting on the space curve determined by

$$S_1 \equiv k_1 x + g_1 = 0 \qquad S_2 \equiv k_2 x + g_2 = 0$$

we can see that the solitons determined by $g_{1\xi}$ and $g_{2\xi}$ interact linearly. That means the only nonlinear effect of the plane solitons is of deforming the shape of the curve.

(b) Two plane soliton solutions. If the functions g_1 and g_2 are fixed as linear in spacetime, i.e. for the plane solitons,

$$g_1 = P_1 y + Q_1 z - (k_1^2 P_1 + k_1^4 Q_1)t + x_{10} \tag{52}$$

$$g_2 = P_2 y + Q_2 z - (k_2^2 P_2 + k_2^4 Q_2)t + x_{20} \tag{53}$$

the constants P_1, P_2, Q_1 and Q_2 are all free while the interaction constant of two solitons should be fixed as

$$\begin{aligned} a_{12} = (k_1 - k_2)[((k_1-k_2)^4 - k_2^4)Q_2 - ((k_1-k_2)^4 - k_1^4)Q_1 + k_1(k_1 - 2k_2)P_2 \\ + k_2(2k_1 - k_2)P_1]\{(k_1+k_2)[((k_1+k_2)^4 - k_2^4)Q_2 + ((k_1+k_2)^4 - k_1^4)Q_1 \\ + k_1(k_1+2k_2)P_2 + k_2(2k_1+k_2)P_1]\}^{-1}. \end{aligned} \tag{54}$$

Correspondingly, the general two plane soliton solutions for the fields w, u and v_1 are

$$w = \frac{k_1 J_1 + k_2 J_2 + (k_1+k_2)a_{12} J_1 J_2}{1 + J_1 + J_2 + a_{12} J_1 J_2} \qquad (J_i \equiv \exp(k_i x + g_i),\, i = 1, 2) \tag{55}$$

$$u = \frac{(a_{12}(k_1+k_2)^2 + (k_1-k_2)^2)J_1 J_2 + k_1^2 J_1 + k_2^2 J_2 + a_{12}k_1^2 J_1 J_2^2 + a_{12}k_2^2 J_2 J_1^2}{(1 + J_1 + J_2 + a_{12}J_1 J_2)^2} \tag{56}$$

$$\begin{aligned} v_1 = \frac{P_1[k_1 J_1 + (k_1 - k_2 + a_{12}(k_1 + k_2 + k_1 J_2))J_2 J_1]}{(1 + J_1 + J_2 + a_{12}J_1 J_2)^2} \\ + \frac{P_2[k_2 J_2 + (k_2 - k_1 + a_{12}(k_2 + k_1 + k_2 J_1))J_2 J_1]}{(1 + J_1 + J_2 + a_{12}J_1 J_2)^2}. \end{aligned} \tag{57}$$

It is clear that in this case, only two plane solitons are included in the solutions. There is not any kind of dromion structure for all fields w, u and v_1.

(ix) *The fourth type of multi-dromion.* For $k_2 = -2k_1$, the functions h and h_1 can only be arbitrary functions of a single variable

$$\xi_1 = z - 5k_1^2 y + 4k_1^4 t \tag{58}$$

while for the functions g_1 and g_2, we have two subcases.

(a) If we select h_1 to be proportional to the inverse of the function h, i.e.

$$h_1 = h_1(\xi_1) = a h^{-1} \qquad (a = \text{arbitrary constant}) \tag{59}$$

then

$$g_1 = g_1(\xi_2) = g_1(z - 10k_1^2 y + 9k_1^4 t) \tag{60}$$

and

$$g_2 = g_2(\xi_3) = g_2((a-5)z - 5k_1^2(a-15)y + k_1^4(4a-70)t) \tag{61}$$

are two arbitrary functions of two different arguments ξ_2 and ξ_3. The corresponding solutions for w, u and v_1 read

$$w = \frac{-k_1 h(-J_1 + 2J_2 + aJ_1 J_2)}{h^2 + h(J_1 + J_2) + aJ_1 J_2} \qquad (J_i \equiv \exp(k_i x + g_i),\, i = 1, 2) \tag{62}$$

$$u = k_1^2 h \frac{(a+9)hJ_1J_2 + h^2(J_1+4J_2) + a(J_1J_2^2 + 4J_2J_1^2)}{(h^2 + h(J_1+J_2) + aJ_1J_2)^2} \tag{63}$$

$$\begin{aligned} v_1 = {} & \frac{k_1^3}{(h^2 + h(J_1+J_2) + aJ_1J_2)^2}\{5(a-15)g_{2\xi_3}[2h^3J_2 + ((a+3)h^2 + 2ahJ_1)J_2J_1] \\ & +10g_{1\xi_2}[-h^3J_1 + ((a-3)h^2 + ahJ_2)J_2J_1] \\ & +5h_{1\xi_1}[h^2(J_1 - 2J_2) - aJ_1J_2(2h - J_2 + 2J_1)]\}. \end{aligned} \tag{64}$$

Now the fourth type of multi-dromion solution can be obtained by selecting the arbitrary functions g_1, g_2 and h for the potential v_1. Because the arguments of the functions g_1, g_2 and h are all different, the fourth type of multi-dromion is driven by three sets of parallel plane solitons and two camber solitons which are also non-parallel to the x-axis. The planes are parallel in the same set and non-parallel in different sets. In this case, the ring type of dromion is not allowed and there is no phase shift involved in the interaction of this type of multi-dromion.

(b)

$$h_1 = h_1(\xi_1) \qquad h = h(\xi_1) \qquad g_1 = g_2 = g = g(\xi_1). \tag{65}$$

This subcase may be considered as a special case of (i) after relaxation of the constraint condition (37) such that h_1 is an arbitrary function of ξ_1 while h can be fixed as unity again.

(x) *The fifth type of multi-dromion.* For $k_2 = -k_1$, we can take $h_1 = 0$ simply. In this case, we find that the function h can also be an arbitrary function of the multi-variables,

$$\xi_i = P_i y + Q_i z - (k_1^2 P_i + k_1^4 Q_i)t + x_{0i} \qquad i = 1, 2, \ldots, M \tag{66}$$

where P_i, Q_i and x_{0i} are arbitrary constants while g_1 and g_2 can only be arbitrary functions of a single variable $\xi_1 = z - 5k_1^2 y + 4k_1^4 t$. The corresponding solutions for the fields w, u and v_1 have the following forms:

$$w = -\frac{k_1 \sinh S}{G + \cosh S} \qquad G \equiv 2h \exp \frac{1}{2}(-g_1 - g_2) \qquad S \equiv k_1 x + \frac{1}{2}(g_1 - g_2) \equiv k_1 x + g \tag{67}$$

$$u = -\frac{k_1^2(G\cosh S + 1)}{(G + \cosh S)^2} \tag{68}$$

$$v_1 = -\frac{5k_1^3 g_{\xi_1}(G\cosh S + 1) + k_1 \sinh S \sum_{i=1}^M P_i G_{\xi_i}}{(G + \cosh S)^2}. \tag{69}$$

The fifth type of multi-dromion solution (69) is driven by one set of parallel plane solitons (which are determined by the arbitrary function g) and the multi-camber solitons. Only one of the camber solitons which is determined by the camber $S = 0$ is non-parallel to the x-axis. Similar to the first type of multi-dromion soliton solution, one set of camber solitons which are given by the selecting of the function G can possess quite free shapes, say, a cylinder shape. Different from the first type of multi-dromion solution, the existence of the camber solitons (which are parallel to the x-axis and determined by the function G) does not deform the shape of the camber S, although the multi-parallel plane solitons (which is determined by the function g) do. Different from all other types of multi-dromion solutions, three non-parallel plane solitons can be used to construct a significant point-like dromion solution. If we take $g = 0$, and G is given by any one of equations (19), (29) and (30), a point-like dromion is obtained. But now the dromions are different from those obtained from the first type of multi-dromion solution. In this case the single dromion solution is anti-symmetric with respect to the camber $S = 0$ because it possesses a different sign for

$S > 0$ and $S < 0$, while the dromion solution of the first type is symmetric with respect to the camber $S = 0$.

Although the multi-camber solitons can be taken as in the first case, say, $\sum_{i=1}^{M} P_i G_{\xi_i} \equiv H_j$, $j = 1, 2, \ldots, 7$, the nonlinear interactions among them are much more complicated than those of in the first case because G appears not only in the numerator of (69), but also in the denominator. The phase shifts among the dromions are dependent on the selected multi-camber soliton solutions, H_7, of the seed models.

2.3. $N \geqslant 3$ case

From the results of the last two subsections, we know that the rich structures of the dromion solutions of equation (1) are strongly dependent on the arbitrariness of the functions appearing in the solutions. This arbitrariness is reduced sharply as N increases. For $N = 1$, the functions are arbitrary with respect to an *arbitrary number* (M) of *arbitrary planes* (no constraints on constants k_1, P_{1i} and Q_{1i}) with arguments ξ_{ij}. When N is increased to $N = 2$, every function is arbitrary only for *single* and *special plane* arguments although the different functions may possess different plane arguments.

The further consideration will show us that the arbitrariness of functions will disappear for $N \geqslant 3$. For $N = 3$, a special three-plane soliton solution with

$$\phi = 1 + \sum_{i=1}^{3} \exp \xi_i + \sum_{j<i}^{3} a_{ij} \exp(\xi_i + \xi_j) + a_{123} \exp(\xi_1 + \xi_2 + \xi_3) \tag{70}$$

$$\begin{aligned} \xi_i &= k_i x + P_i y + Q_i z - (k_i^2 P_i - k_i^4 Q_i)t + \xi_0 \\ a_{ij} &= (k_j - k_i)[((k_j - k_i)^4 - k_i^4)Q_i - ((k_j - k_i)^4 - k_j^4)Q_j + k_j(k_j - 2k_i)P_i \\ &\quad + k_i(2k_j - k_i)P_j]\{(k_j + k_i)[((k_j + k_i)^4 - k_i^4)Q_i + ((k_j + k_i)^4 - k_j^4)Q_j \\ &\quad + k_j(k_j + 2k_i)P_i + k_2(2k_j + k_i)P_j]\}^{-1} \end{aligned} \tag{71}$$

and $a_{123} = a_{12}a_{13}a_{23}$ can exist if an additional constrained condition among constants k_i, P_i and Q_i is satisfied. If we take $Q_i = 0$, i.e. the field is z-independent, the constrained condition reads

$$\begin{aligned} & a_{12}(k_1 + k_2)[(k_2^2 + 2k_1k_2 - 3k_3(k_1 + k_2) + 3k_3^2)P_1 + (k_1^2 + 2k_1k_2 - 3k_3(k_1 + k_2) + 3k_3^2)P_2 \\ &\quad + (-2k_3^2 + 3k_3(k_1 + k_2) - (k_1 + k_2)^2)P_3] \\ &\quad + a_{13}(k_1 + k_3)[(k_3^2 + 2k_1k_3 - 3k_2(k_1 + k_3) + 3k_2^2)P_1 \\ &\quad + (k_1^2 + 2k_1k_3 - 3k_2(k_1 + k_3) + 3k_2^2)P_3 \\ &\quad + (-2k_2^2 + 3k_2(k_1 + k_3) - (k_1 + k_3)^2)P_2] \\ &\quad + a_{23}(k_3 + k_2)[(k_2^2 + 2k_3k_2 - 3k_1(k_3 + k_2) + 3k_1^2)P_3 \\ &\quad + (k_3^2 + 2k_3k_2 - 3k_1(k_3 + k_2) + 3k_1^2)P_2 \\ &\quad + (-2k_1^2 + 3k_1(k_3 + k_2) - (k_3 + k_2)^2)P_1] = 0. \end{aligned} \tag{72}$$

We do not write down the general constrained condition for $Q_i \neq 0$ because of its complexity (over three printed pages).

For $N \geqslant 4$, much more constrained conditions should be satisfied for multi-plane soliton solutions. For instance, there are 20 complicated constrained conditions for 12 parameters k_i, P_i and Q_i of the four-plane soliton solution. These overdetermined constrained equations have no solutions except for the case $P_i = ak_i$ and $Q_i = bk_i$ for all i which corresponds to the fact that equation (1) possesses a $(1 + 1)$-dimensional integrable reduction [14], $u = U(X, t)$, with space variable $X = x + ay + bz$. Usually, an integrable model possesses the Painlevé property and generalized multi-plane soliton solutions [15]. However, for the $(3+1)$-dimensional KdV-type equation (1), we do not have both the Painlevé property and the generalized three- or more-plane soliton solutions. This fact suggests to us that the model (1) may be non-integrable.

3. Summary and discussion

In summary, the dromion-like structure of a $(3+1)$-dimensional model is much more abundant than those of $(2+1)$-dimensional cases. For the KdV-type equation (1), a single dromion may have quite a free shape. For example, after selecting the arbitrary functions appropriately, we can get point-like dromions, ring type dromions, extended and sharp dromions and oscillatory dromions. For the multi-dromion solutions, there are five types of structures.

The first type of multi-dromion is driven by multi-camber ghost solitons. One of the camber solitons is not parallel to the x-axis while the others are all parallel to the x-axis. The point-like dromion is located at the cross point of three cambers while the ring dromion solution is driven by a cylinder soliton (which is parallel to the x-axis) and the camber soliton (which is not parallel to the x-axis) and located at the intersection line of the cylinder and the camber. The multi-soliton solutions of any $(D+1)$-dimensional integrable model can be used to construct the multi-camber soliton solutions which are parallel to the x-axis. The second type of multi-dromion solution is driven by one set of parallel plane solitons and two camber solitons. Only the camber solitons are not parallel to the x-axis. The set of plane solitons is described by three arbitrary functions. All three functions have the same single argument. The third type of multi-dromion is driven by two sets of parallel plane solitons and two camber solitons which are not parallel to the x-axis. Two sets of plane solitons are determined by two arbitrary functions. Every function is a function of a single argument only, but the arguments for different functions are not the same. The fourth type of multi-dromion is driven by three sets of plane solitons and two camber solitons. Three sets of plane solitons are described by three arbitrary functions with three different arguments although every function is only a function of a single variable. The ring type of dromion is not allowed for the second, third and fourth types of solutions.

The fifth type of multi-dromion solution is driven by one set of plane solitons (described by one arbitrary function of a single variable), a set of camber solitons which are parallel to the x-axis (and described by an arbitrary function of multi-arguments) and a camber soliton which is non-parallel to the x-axis and non-symmetric with respect to the camber. The ring type of dromion can also be constructed from the fifth type of solution but the ring dromions in this case are different from those of the first type of solution. The first type of ring dromion solution is symmetric with respect to the camber $S=0$ and the existence of the ring dromion will deform the shape of the camber, while the fifth type of ring dromion solution is antisymmetric with respect to the camber and the existence of the ring dromion will not deform the shape of the camber.

For the first and fifth types of the multi-dromion solutions, some different types of phase shifts may be involved in their interactions because the sets of the camber solitons which are parallel to the x-axis can be selected as the multi-soliton solutions of any $(D+1)$-dimensional integrable seed models. However, for other types of multi-dromion solutions, there is no phase shift involved in their interactions. The only effects of the nonlinear interactions among the camber and plane solitons of the other types of the multi-dromion solutions are to deform the shape(s) of the camber soliton(s) which is(are) non-parallel to the x-axis.

In $(2+1)$-dimensional cases, we have not yet found a ring type of dromion solution. Although the ring type of dromion solution can be found in $(3+1)$-dimensional cases, we have not found a bubble-like dromion solution which is finite at a closed surface and decays away from the surface in $(3+1)$ dimensions.

All the multi-dromion solutions reported here for the KdV-type equation (1) are driven by multi-camber (and plane) solitons. The abundant structure of the dromions is strongly dependent on the existence of the arbitrary functions in the solutions. As N increases to $N \geqslant 3$ this arbitrariness will disappear. Even for the three-plane soliton solution, the planes cannot be arbitrary because the model is non-integrable. If one can find some $(3+1)$-dimensional integrable models, we believe that at least some of the similar dromion solutions can be found because some arbitrary functions should be found in the symmetry algebras of high-dimensional integrable models [8, 9, 16, 17].

The dromion solutions have been found in many physically significant $(2+1)$-dimensional models such as the DS, KP and NNV equations. From the discussions of this paper for the $(3+1)$-dimensional KdV-type toy model, we have seen that the dromion structures can also exist. We hope that future study will find the dromion structures in the real physical $(3+1)$-dimensional models.

Acknowledgments

The author would like to thank the useful comments of the referees and the discussions of Professors Y-s Li, Q-p Liu, X-b Hu and G-j Ni. The work was supported by the Nature Sciences Foundation of China and the Natural Science Foundation of Zhejiang Province of China.

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